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ON THE FORMATION OF SHOCK WAVES IN LAVAL NOZZLES*

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Inviscid gas flows are investigated in the case when in plane and axisymmetric Laval nozzles a weak discontinuity propagates along the characteristic which reaches the nozzle center. Sufficient conditions of shock wave formation at the nozzle center, disclosed by the first correction of the continuous self-similar solution, are obtained.

The inverse problem of the Laval nozzle consists of constructing in some neighborhood of the nozzle longitudinal axis a flow conforming to a given velocity distribution along that axis and, in particular, to determine the shape of the nozzle walls. It must be pointed out that not every velocity distribution is admissible. Frankl' had established /1/ an inequality which determines continuity conditions for velocity distribution along the axis for which acceleration has a finite discontinuity at the nozzle center. It was shown in /2/ that when the discontinuity exceeds the value defined by that inequality, first, a shock flow is generated with a compression shock at the nozzle center and, then at some critical magnitude of the discontinuity, the flow disintegrates. The condition of shock flow existence in plane and axisymmetric nozzles in the class of self-similar solutions of the transonic equation was determined, and it was shown that the reason of shock wave formation is the prolongation of the nozzle transition part when its throat lies downstream of its center.

The effect of higher approximations in axial velocity distribution on the properties of flow are investigated in this paper. It is shown that a shock wave may be formed also in the case when the principal term defines a continuous self-similar flow. Similar situation was earlier considered /3,4/ in the problem of a weak discontinuity reflection from the sonic line in connection with the problem of shock wave generation.

The results obtained in this investigation have a very simple physical meaning: flows in nozzles with acceleration in the direction toward the outlet remain shock-free in the class of functions considered here. The last of continuous self-similar flows in which the disclosure of a shock wave is possible in higher approximations is an exception. As regards flows with local supersonic zones locked on the channel axis (LSZ) they are unstable at small perturbations, and ultimately are easily transformed from shock-free to shock flows.

1. Transonic flows of gas are defined by the Kármán approximate system of equations

$$-uu_x + v_y + \omega y^{-1}v = 0, \quad u_y - v_x = 0 \tag{1.1}$$

where x, y are dimensionless Cartesian or cylindrical coordinates, u, v are dimensionless velocity components of a uniform sonic stream perturbations, and parameter $\omega = 0$ $(\omega = 1)$ in a plane (axisymmetric) flow.

To investigate the motion of gas in a Laval nozzle we set up the Cauchy problem: find the solution of system (1.1) which on the axis of symmetry $y = 0$ satisfies the conditions

$$\begin{aligned} &u(x, 0) = A_0x + A_1|x|^{1+k} + \ldots & x < 0 \\ &u(x, 0) = B_0x + B_1x^{1+k} + \ldots & x > 0 \\ &v(x, 0) = 0 \end{aligned} \tag{1.2}$$

where k, A_i, B_i are arbitrary constants with $A_0 > 0, k > 0$, and /2/

$$r_5 \leqslant B_0/A_0 \leqslant 1 \quad (r_5 = -2(1-\omega) - 2^{-1}(3+\sqrt{5})\,\omega)$$

We denote the limit characteristic which reaches the nozzle center by C_0^-. We investigate below the motion of gas when a weak discontinuity arrives along the characteristic C_0^- in the nozzle center. We denote the region between the negative semiaxis x and characteristic

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C_0^- by the numeral 1. If the flow as no shock wave issuing from the nozzle center, there exists a second limit characteristic C_0^+. The region between characteristics C_0^- and C_0^+ will be denoted by the numeral 2, and the outlet section of the nozzle between C_0^+ and the positive semiaxis x by 3. If a shock wave passes through the channel center, it separates regions 2 and 3.

In the case of shock wave formation functions u, v satisfy at the shock the supplementary boundary conditions /4/

$$2\,(dx/dy)^2 = u_3 + u_2,\ (u_3 - u_2)\,dx/dy = v_2 - v_3 \tag{1.3}$$

where $x = x(y)$ is the equation of the wave front and the subscripts denote the unknown functions on different sides of the shock.

2. Let us present some results from /1,2,5-7/ concerning the particular case of boundary conditions (1.2)

$$u\,(x,\,0) = A_0 x,\ x < 0,\ u\,(x,\,0) = B_0 x,\ x > 0, \qquad v\,(x,\,0) = 0 \tag{2.1}$$

The respective Cauchy problem (1.1), (2.1) has the self-similar solution

$$u = y^2 f_0(\zeta),\ v = y^3 g_0(\zeta),\ \zeta = xy^{-2} \tag{2.2}$$

The substitution of (2.2) into (1.1) yields a nonlinear system of ordinary differential equations

$$(4\zeta^2 - f_0)\,f_0' - 4\zeta f_0 + (3 + \omega)\,g_0 = 0 \qquad 2f_0 - 2\zeta f_0' - g_0' = 0 \tag{2.3}$$

System (2.3) has singular points $\zeta = z_-$ and $\zeta = z_+$ that correspond to the limit characteristics C_0^- and C_0^+. We denote constants z_-, z_+ by z_c. At point $\zeta = z_c$ we have the equality

$$4\zeta^2 - f_0(\zeta) = 0 \tag{2.4}$$

Classification of the various modes of self-similar flow is conveniently carried out using curves of function $f_0(\zeta)$ which show the dependence of velocity u on x along the line $y = \text{const}$ (Fig.1). Flows in regions 1, 2, 3 are represented by branches Q_1, Q_2, Q_3. The letter P denotes the parabola $f_0 = 4\zeta^2$ on which function $f_0(\zeta)$ can have discontinuous derivatives. The points of parabola P are singular points of the node type for system (2.3). Integral curves reach P either at the slope $f_0' = \gamma_1 z_c$ $(\gamma_1 = -4\,(1 - \omega) - (2 + 2\sqrt{5})\,\omega)$ or $f_0' = \gamma_2 z_c$ $(\gamma_2 = 2(1 - \omega) - (2 - 2\sqrt{5})\,\omega)$. The approach of the curve to the parabola at slope $f_0' = \gamma_1 z_c$ is indicated by an arrow.

The structure of the self-similar solution depends on the ratio of constants B_0/A_0.

When $B_0/A_0 = 1$, mode 1 obtains in the nozzle in which the flow is continuous and analytic along limit characteristics, and when $r_3 < B_0/A_0 < 1$ $(r_3 = 4^{-1}(1 - \omega) + 2^{-1}(7 - 3\sqrt{5})\omega)$ we have mode 2 of continuous flow with weak discontinuities on C_0^- and C_0^+.

When $B_0/A_0 = r_3$ we obtain the last of continuous flows (mode 3), and when $0 \leqslant B_0/A_0 < r_3$, a compression shock appears in the flow (mode 4).

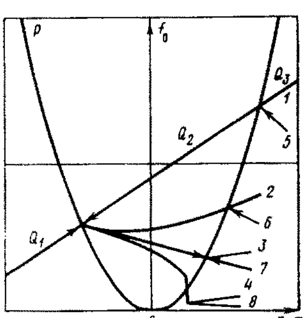

Fig.1

The flows 1-4 are distinguished by that the flow velocity at the nozzle outlet is supersonic and in the limit case of $B_0/A_0 = 0$ it is sonic. In modes 5-8 LSZ appear at the channel wall and merge on the axis.

When $r_5 < B_0/A_0 < r_7$ $(r_7 = -2^{-1}(1 - \omega) - 2^{-1}(3 - \sqrt{5})\,\omega)$ the flow in LSZ is shock-free (mode 6).

Modes 5 $(B_0/A_0 = r_5)$ and 7 $(B_0/A_0 = r_7)$ are limit ones for continuous flow in LSZ.

When $r_7 < B_0/A_0 < 0$ we have mode 8 with the formation of a compression shock in LSZ.

3. Let us construct higher approximations of the self-similar flow (2.2) which would result in velocity distribution (1.2) on the nozzle axis.

If we apply, as in /9/, the conventional coordinate expansion in powers of y with coefficients dependent on ζ, we obtain an increase in the coefficients of singularities in the neighborhood of point $\zeta = z_c$. Such solution is not valid near the limit characteristic C_0. This singularity is explained by the deviation of the non-self-similar characteristic C_0 from the line $\zeta = z_c$.

To obtain a uniformly suitable solution we use the method of deformed coordinates /8/ in which the solution is represented in parametric form

$$u = y^2 [f_0(z) + y^{2k} f_1(z) + y^{4k} f_2(z) + \ldots]$$
$$v = y^3 [g_0(z) + y^{2k} g_1(z) + y^{4k} g_2(z) + \ldots] \tag{3.1}$$
$$\zeta = z + y^{2k}\xi_1(z) + y^{4k}\xi_2(z) + \ldots$$

where z is the new independent variable. The value $z = z_c$ determines the limit characteristic C_0. The deforming functions $\xi_i(z)$ are selected so that singularities in representative velocities $f_i(z)$, $g_i(z)$ do not increase.

To determine f_i, g_i $(i = 1, 2, \ldots)$ it is necessary to solve the recurrent system or ordinary differential equation

$$L_1(f_1, g_1) = (4z^2 - f_0)\xi_1' f_1' + (2 + 2k)(2f_0 - 4zf_0')\xi_1 \tag{3.2}$$
$$M_1(f_1, g_1) = -2f_0\xi_1' - (2 + 2k)\xi_1 f_0'$$
$$L_2(f_2, g_2) = (f_1' - \xi_1' f_0')[(4z^2 - f_0)\xi_1' - (2 + 2k)4z\xi_1 + f_1] + \text{NST}, \quad M_2(f_2, g_2) = \text{NST} \tag{3.3}$$
$$L_i \equiv (4z^2 - f_0) f_i' - [f_0' + 2z(2 + 2ki)] f_i + (3 + \omega + 2ki) g_i$$
$$M_i \equiv -2zf_i' + (2 + 2ki) f_i - g_i'$$

where NST denotes terms which do not induce the growth of singularities in f_2, g_2 as compared to f_1, g_1.

System (3.2), (3.3) has a singular point $z = z_c$, since when $z \to z_c$ the equality (2.4) is valid.

Let us determine the behavior of functions f_1, g_1, ξ_1 in the neighborhood of z_c. Since at that point these functions have discontinuous derivatives, it is necessary to consider separately the left- and right-hand sides of point z_c neighborhood. For definiteenss we shall consider the left half $(z \to z_c - 0)$.

We represent the solution of system (3.2) in the form

$$f_1(z) = F_1(z) + f_0'(z)\xi_1(z), \quad g_1(z) = G_1(z) + g_0'(z)\xi_1(z) \tag{3.4}$$

where F_1, G_1 is the solution of the homogeneous system

$$L_1(F_1, G_1) = 0, \quad M_1(F_1, G_1) = 0 \tag{3.5}$$

Second terms in (3.4) represent the particular solution of the inhomogeneous system (3.2).

We expand function $F_1(z)$ in series with $z \to z_c - 0$

$$F_1(z) = F_{10} + F_{11}\Delta + \ldots + F_{1\mu}|\Delta|^\mu + \ldots, \quad \Delta = z - z_c \tag{3.6}$$

A similar solution holds for $G_1(z)$. The coefficients of series (3.6) linearly depend on constants A_1, B_1 in (1.2), which becomes evident after the transformation of (1.2) into conditions for functions f_1, g_1 as $z \to \pm\infty$. The singularity index μ of the first term of the irregular part is calculated using formulas

$$\mu = (1/2 + 2k/3)(1 - \omega) + \omega[15 - 5\sqrt{5} + 2k(5 - \sqrt{5})]/10 \tag{3.7}$$
$$\text{if } f_0'(z_c - 0) = \gamma_1 z_c$$
$$\mu = (2 + 4k/3)(1 - \omega) + \omega[15 + 5\sqrt{5} + 2k(5 + \sqrt{5})]/10 \tag{3.8}$$
$$\text{if } f_0'(z_c - 0) = \gamma_2 z_c$$

Let us determine the expansion of function $\xi_1(z)$ as $z \to z_c - 0$. Analysis of the right-hand sides of system (3.3) show that the singularities in f_2, g_2 do not increase in comparison with f_1, g_1, if we set

$$(4z^2 - f_0)\xi_1' - (2 + 2k)4z\xi_1 + f_1 = O(\Delta) \tag{3.9}$$

Let us first assume that in expansion (3.6) the singularity index $0 < \mu < 1$. Then from (3.9), (3.4), and (3.6) follows that

$$\xi_1(z) = \xi_{10} + \xi_{1\mu}|\Delta|^\mu + O(\Delta) \tag{3.10}$$
$$\xi_{10} = F_{10}/[4z_c(2 + 2k) - f_0'(z_c - 0)] \tag{3.11}$$
$$\xi_{1\mu} = F_{1\mu}/[6z_c(1 - \omega) + 4\sqrt{5}z_c\omega]$$

Since the coefficients F_{10}, $F_{1\mu}$ are some linear combinations of constants A_1, B_1, hence it follows from (3.11) that the dependence of ξ_{10}, $\xi_{1\mu}$ on A_1, B_1 is also linear.

If $\mu \geqslant 1$, then as $z \to z_c - 0$

$$\xi_1(z) = \xi_{10} + O(\Delta), \ \mu > 1 \tag{3.12}$$

$$\xi_1(z) = \xi_{10} + \xi_{11}\Delta \ln|\Delta| + O(\Delta), \ \mu = 1 \tag{3.13}$$

As shown by the analysis of the uniformly valid solution (3.1), either a finite discontinuity of the first velocity derivatives with respect to coordinates, or a weaker singularity is formed on the limit characteristic C_0.

4. Let us show that the first correction of solution (3.1) can result in the formation of a limit line in the neighborhood of C_0, in spite of the principal self-similar term of (2.2) defining a velocity field without it.

Let us consider the Jacobian of transformation of variables $(\zeta, y) \to (z, y)$

$$J = \partial(\zeta, y)/\partial(z, y) = \partial\zeta/\partial z$$

Taking into account (3.1) we obtain

$$J = 1 + y^{2k}\xi_1'(z) + y^{4k}\xi_2'(z) + \ldots \tag{4.1}$$

We shall investigate J in the neighborhood of C_0, making z approach $z_c - 0$ with $y = \text{const}$.

Let us first consider the case of $0 < \mu < 1$. Substituting (3.10) into (4.1) and taking into account that the singularities of functions ξ_i do not increase, we obtain

$$J = 1 + y^{2k}[-\mu\xi_{1\mu}|\Delta|^{\mu-1} + O(1)] + y^{4k}O(|\Delta|^{\mu-1}) + \ldots$$

Since the index $\mu - 1$ is negative, J becomes infinite on C_0. If $\xi_{1\mu} < 0$, then $J \to +\infty$ and the Jacobian does not change its sign at the respective approach to the limit characteristic.

When $\xi_{1\mu} > 0$, we have $J \to -\infty$, and the Jacobian changes its sign from plus to minus in the neighborhood of C_0, which indicates the formation of a limit line.

If $\mu = 1$, then using (3.13) and (4.1) we obtain

$$J = 1 + y^{2k}[\xi_{11}\ln|\Delta| + O(1)] + y^{4k}O(\ln|\Delta|) + \ldots$$

and the limit line is formed in this case at $\xi_{11} > 0$, while being absent when $\xi_{11} < 0$.

Let us now assume that $\mu > 1$. Then, as $z \to z_c - 0$, function $\xi_1(z)$ is of the form (3.12). From this $\xi_1'(z) = O(1)$ and similarly $\xi_i'(z) = O(1)$, $i = 2, 3, \ldots$ Hence in (4.1) the corrections are small in comparison with the principal term which is equal unity, and the Jacobian does not change its sign on approaching C_0.

Summarizing the above results, we can state that the condition of formation of the limit line indicated by the first correction is

$$0 < \mu \leqslant 1, \ \xi_{1\mu} > 0 \tag{4.2}$$

Analysis of the equation of the limit line $J = 0$ passes through the nozzle center.

5. Inequality (4.2) determines some condition imposed on the quantities k, A_1, B_1 of the Cauchy data (1.2), which is sufficient for generating in the stream of limit line.

Let us first assume that in the Cauchy data (1.2) the exponent k assumes the following values:

$$k > k_* \ (k_* = (1 - \omega)\,3/4 + \omega\sqrt{5}/2) \tag{5.1}$$

It then follows from (5.1), (3.7), and (3.8) that $\mu > 1$ and the limit lines is not formed when $z \to z_c - 0$.

Let now k lies in interval $0 < k \leqslant k_*$.

If in the self-similar solution $f_0'(z_c - 0) = \gamma_2 z_c$, then index μ is calculated by formula (3.8) and we have $\mu > 1$. If $f_0'(z_c - 0) = \gamma_1 z_c$, then μ is calculated by formula (3.7) and satisfies the inequality

$$(1 - \omega)/2 + \omega(15 - 5\sqrt{5})/10 < \mu \leqslant 1$$

Thus, when exponent $0 < k \leqslant k_*$ in the boundary condition (1.2), the derivative of the self-similar solution is $f_0'(z_c - 0) = \gamma_1 z_c$ and condition $\xi_{1\mu} > 0$ (some inequality for A_1, B_1) is satisfied by the first correction, then, as $z \to z_c - 0$ a limit line is formed in the neighborhood of the characteristic C_0.

A similar sufficient condition of limit line formation is valid also when $z \to z_c + 0$ (with the proviso that for $0 < k < k_*$ the inequality $\xi_{1\mu} < 0$) is substituted for $\xi_{1\mu} > 0$.

6. Let us investigate perturbations of the several flow modes shown in Fig.1, where the arrows denote approaches to parabola P with $f_0'(z_c \pm 0) = \gamma_1 z_c$. Conclusions of the preceding Sect. enable us to consider these approaches as dangerous, since then the first approximation may disclose the appearance of a limit line.

Here and subsequently we assume that $0 < k < k_*$. The case of $k = k_*$, when a logarithmic singularity propagates along C_0, although qualitatively similar to the one considered here, requires separate investigation.

When condition $\xi_{1\mu} > 0$ is satisfied by the first correction, a limit line appears when $z \to z_- - 0$ in region 1. Let us rewrite the inequality $\xi_{1\mu} > 0$ using constant A_1 in the Cauchy data (1.2).

Eliminating from system (3.5) function $G_1(z)$, we obtain for $F_1(z)$ the second order linear differential equation

$$(4z^2 - f_0)F_1'' - [2z(1 + \omega + 4k) + 2f_0'] F_1' + [(2 + 2k)(1 + 2k + \omega) - f_0''] F_1 = 0 \qquad (6.1)$$

We pass in (6.1) to the new independent variable τ /9/

$$z = 8^{-1}(1 + R) A_0 - 4^{-1} R A_0 \tau, \quad R = 3(1 - \omega) + \omega\sqrt{5} \qquad (6.2)$$

For F_1 we obtain the hypergeometric equation with parameters $a = -1 - k$, $b = -{}^1/_2 - \omega/2 - k$, $c = -R^{-1}[4^{-1}(1 - R)(1 + \omega + 4k) + 2]$ /10/.

Formula (6.2) implies that the negative semiaxis x is determined by $\tau = +\infty$ and the characteristic C_0^- by $\tau = 1$. When $z \to z_- - 0$, the variable $\tau \to 1 + 0$. In conformity with the boundary condition (1.2) along the axis with $\tau \to \infty$ we take function F_1 in the form

$$F_1 = A_1 (\tau A_0 R/4)^{-a} F(a, a + 1 - c, a + 1 - b; \tau^{-1}) \qquad (6.3)$$

The analytic continuation of solution (6.3) to the neighborhood of $\tau = 1$ is of the form /10/

$$F_1 = A_1 (\tau A_0 R/4)^{-a} [D_1 F(a, a + 1 - c, 1 - \mu; 1 - \tau^{-1}) + D_2 (1 - \tau^{-1})^{\mu} F(1 - b, c - b, 1 + \mu; 1 - \tau^{-1})] \qquad (6.4)$$

$$D_1 = \Gamma(a + 1 - b)\Gamma(\mu)/[\Gamma(1 - b)\Gamma(c - b)], \quad D_2 = \Gamma(a + 1 - b)\Gamma(-\mu)/[\Gamma(a)\Gamma(a + 1 - c)] \qquad (6.5)$$

where $\mu = c - a - b$, which is consistent with (3.7). Using (6.4) we obtain the expansion of F_1 when $\tau \to 1 + 0$

$$F_1 = A_1 (A_0 R/4)^{-a} [D_1 + D_2 (\tau - 1)^{\mu} + O(\tau - 1)]$$

Comparing it with (3.6) and (3.11) we determine the coefficient $F_{1\mu}$ and then $\xi_{1\mu}$

$$\xi_{1\mu} = A_1 (A_0 R/4)^{-\mu - a} D_2 / [(6 - 6\omega + 4\sqrt{5}\omega) z_-]$$

The analysis of formulas (6.5) shows that in a plane flow the coefficient $D_2 < 0$ for any k from the interval $0 < k < {}^3/_4$. In an axisymmetric flow $D_2 < 0$ for $0 < k < 1$, $D_2 > 0$ for $1 < k < \sqrt{5}/2$, and $D_2 = 0$ for $k = 1$. Moreover we take into account that $z_- < 0$. The sufficient condition of limit line formation in region 1 can be expressed in the form

$$A_1 > 0, \; 0 < k < {}^3/_4 \; (\omega = 0), \qquad A_1 > 0, \; 0 < k < 1; \; A_1 < 0, \; 1 < k < \sqrt{5}/2 \; (\omega = 1) \qquad (6.6)$$

If $A_1 = 0$, then in region 1 $F_1 \equiv G_1 \equiv 0$, which implies that the question of limit lines depends on the following approximation. When $k = k_*$ a logarithmic singularity reaches the nozzle center. When $k = 1$ the coefficient $D_2 = 0$, hence the flow is analytic in region 1, including C_0^-, i.e. the limit line is not generated.

Calculations show that a limit line which reaches the nozzle center cannot be eliminated by a compression shock. Hence conditions (6.6) define an inadmissible Cauchy problem.

7. Let us investigate modes 1 – 4 of nozzle operation with supersonic outlet velocity. As shown in Fig.1, the limit line in mode 1 can be formed in region 2 when $z \to z_- + 0$. Carrying out the same calculations, as in the preceding Sect., we conclude that the line of infinite accelerations appears under conditions

$$(TE)B_1 + (HQ)A_1 > 0, \; 0 < k < k_* \qquad (7.1)$$

$$E = \frac{\Gamma(c)\Gamma(-\mu)}{\Gamma(a)\Gamma(b)} \left[1 - \frac{\sin\pi(c - a)\sin\pi(c - b)}{\sin\pi a \sin\pi b}\right]$$

$$Q = \Gamma(-\mu)\,\Gamma(1-a)\,\Gamma(1-b)/[\Gamma(1+a-c)\,\Gamma(1+b-c)\,\Gamma(\mu)]$$

$$H = \Gamma(a+1-b)\,\Gamma(\mu)/[\Gamma(1-b)\,\Gamma(c-b)]$$
$$T = \Gamma(a+1-b)\,\Gamma(1-c)/[\Gamma(a+1-c)\,\Gamma(1-b)]$$

It is possible to ascertain that (7.1) defines an inadmissible Cauchy problem with perturbations of the self-similar mode 1, since then the limit line reaches the nozzle center. As shown in Fig.1, the limit line does not appear in mode 2 when $z \to z_- + 0,\ z_+ - 0,\ z_+ + 0$.

In mode 3 the line of infinite accelerations can be formed in region 2 when $z \to z_+ - 0$, which takes place when condition

$$(A_0^{-a} HEK_-)A_1 + (B_0^{-a} TQK_+)B_1 > 0,\ 0 < k < k_* \tag{7.2}$$
$$K_{\pm} = [4z_{\pm}(2+2k) - f_0'(z_{\pm} \mp 0)]/[4z_{\pm}(2+2k) - f_0'(z_{\pm} \pm 0)]$$

is satisfied. In that case the limit line issues from the nozzle center. Calculations show that it can be eliminated by a compression shock. An example of flow in a Laval nozzle with a shock wave disclosed by the correction of the continuous self-similar solution is thus obtained.

Consequently flows with acceleration in the direction of the nozzle outlet part (modes 1, 2) remain shock-free when perturbations are small, except in the limit mode 3 with discontinuities of first derivatives of velocity components with respect to coordinates on both characteristics passing through the nozzle center.

Let us consider the perturbation of the self-similar mode 4. As shown in Fig.1, the limit line cannot appear when $z \to z_- + 0$.

Let us investigate modes $5-9$ of flows with local supersonic zones joined on the nozzle axis. The limit line in mode 5 is formed in region 2 when $z \to z_- + 0$ if the condition

$$(A_0^{-a}\ QH)A_1 + [(-B_0)^{-a} HEK_+]\,B_1 > 0,\ 0 < k < k_* \tag{7.3}$$

is satisfied.

Infinite accelerations appear also in region 3 when $z \to z_+ + 0$ if

$$B_1 > 0,\ 0 < k < {}^3/_4\ (\omega = 0) \tag{7.4}$$
$$B_1 > 0,\ 0 < k < 1;\ B_1 < 0,\ 1 < k < \sqrt{5}/2\ (\omega = 1)$$

Calculations show that condition (7.3) defines an inadmissible Cauchy problem (1.2), and inequalities (7.4) represent the conditions of compression shock formation.

As shown in Fig.1 the limit line in mode 6 can appear in region 3 when $z \to z_+ + 0$. The sufficient conditions for its formation are defined by inequalities (7.4). It can be eliminated by the introduction of a shock wave.

If the self-similar mode 7 is perturbed, the limit line may appear in region 3 when $z \to z_+ + 0$. The sufficient conditions of its formation are defined by inequalities (7.4), and infinite accelerations in region 2 occur when $z \to z_+ - 0$ and the inequalities

$$(A_0^{-a} HEK_-)A_1 + [(-B_0)^{-a} QH]\,B_1 > 0,\ 0 < k < k_* \tag{7.5}$$

are satisfied.

When at least one of conditions (7.4) or (7.5) is satisfied, a shock wave appears in the flow.

The condition $r_5 \leqslant B_0/A_0 \leqslant r_7$ of shock-free self-similar flows in nozzles with LSZ obtained in /2/ is, thus, insufficient in the case of non-self-similar flow with velocity distribution along the axis of the form (1.2) for $0 < k \leqslant k_*$. Since perturbation of any self-similar mode $5-7$ may result in the formation of a compression shock (for which fulfillment of (7.4) is sufficient) hence in non-self-similar LSZ flows joined on the nozzle axis, a shock wave may appear independently of whether the position of the channel throat is down-stream of its center or not.

8. Let us show how to construct the shock wave generated by the correction of the continuous self-similar solution (2.2). For definiteness we shall consider perturbation of the self-similar mode 6 under conditions (7.4).

In the uniformly valid solution (3.1) u, v, ζ are functions of variables z, y. Let us consider function $\zeta(z, y)$ in the neighborhood of characteristic C_0^+, using expansion (3.1), (3.10), and (3.12). We have

$$\zeta = z + y^{2k}[\xi_{10} + O(\Delta)] + \ldots \quad z \to z_+ - 0 \tag{8.1}$$
$$\zeta = z + y^{2k}[\xi_{10} + \xi_{1\mu}\Delta^{\mu} + O(\Delta)] + \ldots \quad z \to z_+ + 0 \tag{8.2}$$

where $(\Delta = z - z_+)$.

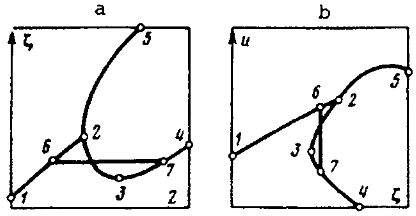

Fig.2

The dependence of ζ on variable z at some small fixed y is qualitatively shown in Fig.2a in the form of curves. Line 1—6—2 represents function $\zeta(z, y)$ defined by (8.1). Curve 2—5 corresponds to function $\zeta(z, y)$ in (8.2) when $\xi_{1\mu} > 0$ and no limit line is formed in the neighborhood of C_0^+. The curve 2—3—7—4 represents form of function (8.2) when $\xi_{1\mu} < 0$, point 3 corresponds to the limit line, and point 2 represents the limit characteristic C_0^+. Segment 6—7 represents the compression shock which in the case of $\xi_{1\mu} < 0$ separates the region of three-valuedness. The curve in Fig.2,b shows the qualitative dependence of the longitudinal component of perturbation velocity u on variable ζ at fixed y.

Prior to proceeding with the construction of the compression shock, we seek an equation of the limit line of the form $z = z_l(y)$. Since it differs from characteristic C_0^+ by higher order smalls, $z_l(y)$ can be expanded in series in powers of y with z_+ as the principal term

$$z = z_l(y) = z_+ + y^{\alpha} l + \ldots \tag{8.3}$$

where α, l are the unknown constants, for whose determination we substitute it into the limit line equation $\partial\zeta/\partial z = 0$. We obtain $\alpha = 2k/(1-\mu)$, $l = (-\mu\xi_{1\mu})^{1/(1-\mu)}$.

If in the uniformly valid solution (3.1) the exponent $2ki$ of y is smaller than α, the respective coefficients at y^{2ki}: $f_i(z)$, $g_i(z)$, $\xi_i(z)$ must be continuous at point $z = z_+$. For this order the boundary of regions 2 and 3 remains the characteristic C_0^+. If exponent $2ki$ in (3.1) satisfies the inequality $2ki > \alpha$, then regions 2 and 3 must be connected by a comparession shock in order to eliminate the limit line.

The forward front of the shock is represented in Fig.2 by point 6 whose equation is of the form

$$z = z_2(y) = z_+ + y^{\alpha} s_2 + \ldots \tag{8.4}$$

where s_2 is to be determined.

The rear front of the shock is represented in Fig.2 by point 7 and is defined in the form of expansion

$$z = z_3(y) = z_+ + y^{\alpha} s_3 + \ldots \tag{8.5}$$

where s_3 is the unknown constant.

Substituting (8.4) and (8.5) into (3.1) we obtain expressions for functions u and v at the shock wave forward and rear fronts. Then, using the shock transition (1.3), we obtain for the unknown s_2, s_3 the system of equations

$$s_2 = s_3 + \xi_{1\mu} s_3^{\mu}, \quad \mu s_2 + s_3 + \mu\xi_{1\mu} s_3^{\mu} = 0$$

whose solution is

$$s_3 = [-2\mu(1+\mu)^{-1}\xi_{1\mu}]^{1/(1-\mu)}, \quad s_2 = (\mu - 1)(2\mu)^{-1} s_3$$

To construct the shock wave with terms higher than y^{α} taken into account it is necessary to include in expansions (3.1) in region 3 behind the shock the terms $y^{\alpha+2k}, y^{\alpha+4k}, \ldots$. The calculation of the velocity jump yields

$$u_3 - u_2 = -[6(1-\omega) + 4\sqrt{5}\omega] z_+ s_2 y^{2+\alpha} + O(y^{2+\alpha+2k})$$

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